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# Inelastic nucleon-deuteron scattering. I. The transition matrix in the impulse approximation for general two-body interactions

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ABSTRACT

Expressions for the transition matrix elements are derived in the impulse approximation, assuming general two-body interactions.

## I. Introduction

The inelastic scattering of nucleons by deuterons has been treated earlier by several authors. Wu and Ashkin [1] and Bransden [2] have formulated the solution of the problem in the Born approximation for various mixtures of ordinary and exchange forces, including also the tensor force. Of these authors, only Bransden takes the tensor force into account in the numerical calculations. In an attempt to find a procedure, more easy to justify for the scattering of nucleons by light nuclei than the Born approximation, Chew [3] proposed the so called impulse approximation and applied it to neutron-deuteron scattering, considering central forces only [4]. Some numerical results for the energy spectrum and angular distribution of neutrons in $n-d$ scattering were also obtained by Gluckstern and Bethe [5], who used a modification of the Born approximation to investigate the importance of interference terms in the total $n-d$ cross section. Franck and Gammel [6] have studied low energy inelastic nucleon-deuteron scattering in the Born approximation, assuming a zero range potential for the nucleon-nucleon interaction.

The point of main interest in the study of the scattering of nucleons by deuterons concerns the conclusions which can be drawn with regard to nucleon-nucleon scattering, the bound and continuum states of the deuteron, and the possible existence of three-body forces. From this point of view the conventional kind of scattering experiments, where only one outcoming particle is registered, are not very conclusive, because they involve, for high energies, an average over a large energy interval and hence a great many states for the two particles not observed. A study of the $d(p, 2p)n$ reaction at high energies would give a much more direct information. Such experiments will, presumably, be made in a near future, so we have considered it of interest to give a theoretical discussion under rather general assumptions about the two-body interactions. At energies above 100 MeV the impulse approximation provides a natural starting point, and by adding correction terms to this approximation one should expect a rather high accuracy.

In this paper we are entirely concerned with a discussion of the transition matrix for the impulse approximation, assuming a general type of necleon-nucleon interaction. Part II will be devoted to an examination of the multiple scattering corrections to the conventional impulse approximation, and will provide a basis for a discussion of the physical content of the formulae derived in the parts I and II. In a subsequent paper some numerical applications will be reported.

## II. General considerations

As a basis for our treatment and in order to introduce concepts and notations we start by giving some formulae from the theory of scattering [7, 8].

We consider a quantum mechanical system consisting of two parts, which collide at the time $t=0$, after having been at infinite separation at $t=-\infty$ in a state $|\phi_i)$. The Hamiltonian $H=K+V$, where $K$ is the Hamiltonian in the absence of any interaction between the two parts and $V$ denotes the interaction. For the calculation of cross sections we introduce the transition operator $T=V\,\Omega^{(+)}$, where $\Omega^{(+)}$, the wave operator introduced by Møller, is such that

$$\Omega^{(+)}\,|\phi_i) = \Psi_i^{(+)}) \tag{1}$$

is the state vector of the system at $t=0$ developing from the initial state $|\phi_i)$. We shall later make use of an analogous operator $T^{(-)}=V\,\Omega^{(-)}$ where

$$\Omega^{(-)}\,|\phi_f) = \Psi_f^{(-)})$$

is that state of the system at $t=0$, which would develop into a final state $|\phi_f)$ at $t=-\infty$, consisting of $|\phi_f)$ plus ingoing waves.

The processes that we wish to study are only those which lead to a disintegration of the deuteron. The final state at $t=\infty$ will then describe three free nucleons. The momenta of two of them determine the final state except for spin. We define, accordingly, a dissociation cross-section $\sigma_{12}^d$ such that[1]

(incident intensity) $\sigma_{12}^d\, d^3k_1^f\, d^3k_2^f =$ the number of processes per unit time in which the incident particle 1 goes into $d^3k_1^f$ around $\mathbf{k}_1^f$ while one of the particles in the deuteron, e.g. 2, goes into $d^3k_2^f$ around $\mathbf{k}_2^f$. (2)

The complete differential cross-section describing the process in which the incident particle 1 goes into $d^3k_1^f$ around $\mathbf{k}_1^f$ is obtained by integrating over particle 2.

$$\sigma_{12} = \int d^3k_2\, \sigma_{12}^d \tag{3}$$

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[1] For simplicity we treat the three particles as distinguishable and introduce the antisymmetry requirement later.

The cross section $\sigma_{12}^{d}$ is related to the matrix elements of the $T$ operator in a well-known manner. We are here interested in scattering at energies $>100$ MeV for which the impulse approximation provides a fairly accurate and simple approximation for the $T$ operator according to

$$T_{1d} = T_{12} + T_{13} \tag{4}$$

Here $T_{1k}$ is the transition operator for collision between the incident particle 1 and particle $k$ in the deuteron ($k = 2, 3$).

### III. Expansion of $T_{1d}$ in spin eigenstates

The formulas for central two-body interactions have been given by Chew [3]. Here we wish to consider more general two-body interactions and have to derive the corresponding formulae which are essentially more complicated. For simplicity we shall only indicate the spin quantum numbers in this section, which means that the matrices are to be regarded as operators in momentum or position space.

As a basis set we consider the four quartet functions with spin 3/2, which are totally symmetric in all coordinates, while the others form two doublets of which one is chosen symmetric and the other antisymmetric in a given pair of coordinates (for definiteness particles 2 and 3).[1]

The doublet functions transform according to the following representation of the permutation operators (12), (13) and (23):

$$(12) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}; \quad (13) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}; \quad (23) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

These operations can be used for transformations to functions with symmetry/antisymmetry in other pairs of particles. Using (5) one finds easily the following orthogonal matrices, written in tabular form for the sake of clarity:

*Table I.* The quantities $(S, s_{23} | S, s_{12})$ for $S = 1/2$.

| | $s_{12} = 0$ | $s_{12} = 1$ |
|---|---|---|
| $s_{23} = 0$ | $-1/2$ | $-\sqrt{3}/2$ |
| $s_{23} = 1$ | $+\sqrt{3}/2$ | $-1/2$ |

*Table II.* The quantities $(S, s_{23} | S, s_{13})$ for $S = 1/2$.

| | $s_{13} = 0$ | $s_{13} = 1$ |
|---|---|---|
| $s_{23} = 0$ | $+1/2$ | $+\sqrt{3}/2$ |
| $s_{23} = 1$ | $\sqrt{3}/2$ | $-1/2$ |

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[1] See, e.g., Verde, reference [9].

It is convenient to express the matrix elements of $T_{1d}$ in spin states with definite symmetry properties in a given pair of particles (e.g. 2 and 3). We have then to consider matrix elements of the form

$$(S^f,\ M^f,\ S^f_{23}\,|\,T_{1d}\,|\,S^i,\ M^i,\ S^i_{23}). \tag{6}$$

As $T_{1d}=T_{12}+T_{13}$ it is convenient to expand the matrix element of $T_{1k}$ ($k=2, 3$) in spin states which are symmetric/antisymmetric in particles 1 and $k$. We then obtain

$$\begin{aligned}(S^f, M^f, s^f_{23}\,|\,T_{1d}\,|\,S^i, M^i, s^i_{23}) = \sum_{k=2}^{3}\ \sum_{\substack{S',\,M',\,s'_{1k}\\ S,\,M,\,s_{1k}}} & (S^f,\ M^f,\ s^f_{23}\,|\,S',\ M',\ s'_{1k})\times\\ &\times(S,\ M,\ s_{1k}\,|\,S^i,\ M^i,\ s^i_{23})\,(S',\ M',\ s'_{1k}\,|\,T_{1k}\,|\,S,\ M,\ s_{1k}).\end{aligned} \tag{7}$$

The quantities $(S^f\,M^f\,s^f_{23}\,|\,S'\,M'\,s_{1k})$ etc. are themselves matrix elements of permutation operators and are therefore diagonal in $S$ and independent of $M$. For the quartets they are equal to 1 and for the doublets we use tables I and II. We can then omit the projection quantum numbers and write (7) as

$$\begin{aligned}(S^f, M^f, s^f_{23}\,|\,T_{1d}\,|\,S^i, M^i, s^i_{23}) = \sum_{k}\ \sum_{\substack{s'_{1k}\\ s_{1k}}} & (S^f,\ s^f_{23}\,|\,S^f,\ s'_{1k})\,(S^i,\ s^i_{23}\,|\,S^i,\ s_{1k})\times\\ &\times(S^f,\ M^f,\ s'_{1k}\,|\,T_{1k}\,|\,S^i,\ M^i,\ s_{1k}).\end{aligned} \tag{8}$$

Table III in the appendix gives the matrix elements of $T_{1d}$ between spin states symmetric/antisymmetric in particles 2 and 3.

Expanding the spinfunctions in the matrix elements of $T_{1k}$ in a Clebsch-Gordan series[1] with $s_{1k}=s, m_{1k}=m$.

$$\begin{aligned}(S',\ M',\ s'\,|\,T_{1k}\,|\,S,\ M,\ s) &=\\ &=\sum_{\substack{m'=-s'\\ m=-s}}^{s',\,s} C(s',\ 1/2,\ S';\ m',\ M'-m')\,C(s,\ 1/2,\ S;\ m, M-m)\times\\ &\qquad\times(s',\ m'\,|\,T_{1k}\,|\,s,\ m)\ \delta_{M'-m',\,M-m}\end{aligned} \tag{9}$$

where we have used that $T_{1k}$ does not depend on the third particle. This expression is formally symmetric in $S, M, s$ and $S', M', s'$ so that the matrix element with the right and the left spin states interchanged is obtained without separate calculation.

A further simplification is obtained noting that for interactions conserving parity $T_{1k}$ is diagonal in $s_{1k}$, so that (9) simplifies to

$$\begin{aligned}(S',\ M',\ s'\,|\,T_{1k}\,|\,S,\ M,\ s) &=\\ &=\delta_{s's}\sum_{m=-s}^{s} C(s,\ 1/2,\ S';\ M'-M+m,\ M-m)\,C(s,\ 1/2,\ S;\ m,\ M-m)\times\\ &\qquad\times(s,\ M'-M+m\,|\,T_{1k}\,|\,s,\ m).\end{aligned} \tag{10}$$

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[1] The notation for the Clebsch-Gordan coefficients is that of Rose [10].

Table IV in the appendix gives the non-vanishing matrix elements of $T_{1k}$ assuming $s_{1k}$ to be conserved.

## IV. Specification of initial and final states

In the preceding section we only studied the dependence on spin quantum numbers. Physically we have to deal with a collision exciting the deuteron to a continuum state characterized by the momenta $\mathbf{p}_2^f = \hbar\,\mathbf{k}_2^f$ and $\mathbf{p}_3^f = \hbar\,\mathbf{k}_3^f$ and a spin state. As measurements of the spin are made in the asymptotic region, it is natural to ask for the transition probabilities to states with welldefined spin quantum numbers. For a non-central interaction between particles 2 and 3 those are not necessarily the same as immediately after the collision, i.e. the spin states which evolve from the initial state by the operators $T_{12}$ and $T_{13}$. To obtain the matrix element for the final spin state, we therefore have to consider the matrix elements for transitions via different intermediate states and then sum over all these.

In accordance with these remarks we compute the matrix elements between the initial state

$$|S^i, M^i, s_{23}^i)\,(S^i, M^i, s_{23}^i\,|\,\mathbf{k}_1^i, \Phi_0) \overset{\text{def}}{=} |\,i) \tag{11}$$

where $\mathbf{k}_1^i$ is the wave vector of the incident particle and $\Phi_0$ is the deuteron ground state wave function in the laboratory system, and the final state

$$|S', M', s'_{23})\,(S', M', s'_{23}\,|\,\mathbf{k}_1^f, \Psi_{\mathbf{k}_2^f, \mathbf{k}_3^f}, S^f, M^f, s_{23}^f) \overset{\text{def}}{=} |\,f) \tag{12}$$

where $\mathbf{k}_1^f$ is the outgoing wave vector for particle 1,

$$|\,\mathbf{k}_1^f, \Psi_{\mathbf{k}_2^f, \mathbf{k}_3^f}, S^f, M^f, s_{23}^f)$$

is a state of definite momentum for particle 1 times a continuum state for particles 2 and 3 in the laboratory system with asymptotic wave vectors $\mathbf{k}_2^f$ and $\mathbf{k}_3^f$ and the asymptotic spin state $|\,S^f, M^f, s_{23}^f)$. $|\,S', M', s'_{23})$ is the intermediate state mentioned above.

We observe that the continuum wave function $|\,\Psi_{\mathbf{k}_2^f, \mathbf{k}_3^f})$ can be written using the wave operator $\Omega_{23}^{(-)}$, as

$$|\,\Psi^{(-)}_{\mathbf{k}_2^f, \mathbf{k}_3^f}) = \Omega_{23}^{(-)}\,|\,\mathbf{k}_2^f, \mathbf{k}_3^f). \tag{13}$$

Since $\Omega_{ij}^{(-)}$ is diagonal in $s_{ij}$, we can use the expansion (10), and therefore table IV, for calculating matrix elements of $\Omega_{ij}^{(-)}$ between three-particle-states.

From our discussion it is clear how the matrix elements of $T_{1d}$ in spin-momentum space can be reduced to operators in momentum space. We regard in the following these operators ($T_{1k}$ and $\Omega_{23}^{(-)}$) as known. We then have to consider the matrix elements of $T_{1d}$ between the initial and final states (11) and (12) and simplify them as far as possible.

## V. Simplification of the matrix elements

In the next few sections we omit the spin quantum numbers in our notations, understanding that the functions $|i)$ and $|f)$ are those defined be equations (11) and (12), regarded as ket-vectors with respect to operators in momentum or position space, while $T_{1k}$ $(k=d,\,1,\,2)$ in the following stands for the matrix elements.

$$(S',\, M',\, s'_{23}\,|\,T_{1k}\,|\,S^i,\, M^i,\, s^i_{23}).$$

We expand $(f\,|\,T_{12}\,|\,i)$ and $(f\,|\,T_{13}\,|\,i)$ in momentum eigenstates:

$$(f\,|\,T_{12}\,|\,i)=\int d^9k'\,d^9k\,(f\,|\,\mathbf{k}'_1\,\mathbf{k}'_2\,\mathbf{k}'_3)\,(\mathbf{k}'_1\,\mathbf{k}'_2\,\mathbf{k}'_3\,|\,T_{12}\,|\,\mathbf{k}_1\,\mathbf{k}_2\,\mathbf{k}_3)\,(\mathbf{k}_1\,\mathbf{k}_2\,\mathbf{k}_3\,|\,i).$$

Introducing the relative and total momenta for particles 2 and 3, defined by

$$\begin{aligned}\mathbf{k}_{23}&=(\mathbf{k}_2-\mathbf{k}_3)/2\\ \mathbf{K}_{23}&=\mathbf{k}_2+\mathbf{k}_3\end{aligned}\tag{14}$$

we get

$$(\mathbf{k}_1\,\mathbf{k}_2\,\mathbf{k}_3\,|\,i)=\delta\,(\mathbf{k}^i_1-\mathbf{k}_1)\,\delta\,(\mathbf{K}^i_{23}-\mathbf{K}_{23})\,g_0\,(\mathbf{k}_{23})\tag{15}$$

where $g_0\,(\mathbf{k}_{23})$ in the Fourier transform of the deuteron wave function in relative coordinates.

Similarly

$$(\mathbf{k}'_1\,\mathbf{k}'_2\,\mathbf{k}'_3\,|\,f)=\delta\,(\mathbf{k}^f_1-\mathbf{k}'_1)\,\delta\,(\mathbf{K}^f_{23}-\mathbf{K}'_{23})\,g_{\mathbf{k}^f_{23}}\,(\mathbf{k}'_{23})\tag{16}$$

where $g_{\mathbf{k}^f_{23}}\,(\mathbf{k}'_{23})$ is the Fourier transform of a continuum wave function for particles 2 and 3 in relative coordinates with asymptotic form $|\,\mathbf{k}^f_{23})$. We note that $g_{\mathbf{k}^f_{23}}\,(\mathbf{k}'_{23})$ has a delta function singularity for $\mathbf{k}'_{23}=\mathbf{k}^f_{23}$.

Analogously, the matrix element of $T_{12}$ is simplified by using relative coordinates for particles 1 and 2, and we obtain

$$(\mathbf{k}'_1\,\mathbf{k}'_2\,\mathbf{k}'_3\,|\,T_{12}\,|\,\mathbf{k}_1\,\mathbf{k}_2\,\mathbf{k}_3)=\delta\,(\mathbf{k}'_3-\mathbf{k}_3)\,\delta\,(\mathbf{K}'_{12}-\mathbf{K}_{12})\,(\mathbf{k}'_{12}\,|\,t_{12}\,|\,\mathbf{k}_{12})\tag{17}$$

where $t_{12}$ is the transition operator in relative coordinates. Assuming the deuteron to be at rest before the collision we get

$$\begin{aligned}(f\,|\,T_{12}\,|\,i)=\int d^3k'_{23}\,g^*_{\mathbf{k}^f_{23}}\,(\mathbf{k}'_{23})\,g_0\,(\mathbf{k}'_{23}-\mathbf{K}^f_{23}/2)\,\delta\,(\mathbf{k}^f_1+\mathbf{k}^f_2+\mathbf{k}^f_3-\mathbf{k}^i_1)\times&\\ \times\left(\frac{\mathbf{k}^f_1-\mathbf{K}^f_{23}/2-\mathbf{k}'_{23}}{2}\,\middle|\,t_{12}\,\middle|\,\frac{\mathbf{k}^i_1+\mathbf{K}^f_{23}/2-\mathbf{k}'_{23}}{2}\right).&\end{aligned}\tag{18}$$

The matrix element of $t_{12}$ is under fairly general conditions, which have been discussed by Chew [3], rather weakly dependent on $\mathbf{k}'_{23}$. One can therefore with good approximation replace it by a constant value, substituting for $\mathbf{k}'_{23}$ the momentum for which the other factors in the intergrand of (18) have the largest value, i.e. $\mathbf{k}^f_{23}$. This may be regarded as the first term in an expansion of

$(f|T_{12}|i)$ the other terms of which we shall not discuss in this part. We shall return to this question in connection with a physical discussion in part II. Using the relations (14) we finally get

$$(f|T_{12}|i) = \delta(\mathbf{k}_1^f + \mathbf{k}_2^f + \mathbf{k}_3^f - \mathbf{k}_1^i)\, M(12)\, I_{23} \tag{19}$$

where

$$\begin{aligned} M(12) &= \left(\frac{\mathbf{k}_1^f - \mathbf{k}_2^f}{2}\Big| t_{12} \Big| \frac{\mathbf{k}_1^i + \mathbf{k}_3^f}{2}\right) \\ I_{23} &= \int d^3k'_{23}\, g^*_{\mathbf{k}'^f_{23}}(\mathbf{k}'_{23})\, g_0(\mathbf{k}'_{23} - \mathbf{K}^f_{23}/2). \end{aligned} \tag{20}$$

The matrix elements of $T_{13}$ can be computed in the same way. The expressions may also be obtained from (19) and (20) by interchanging the indices 2 and 3, noting, however, that $\mathbf{k}'_{23}$ and $\mathbf{k}^f_{23}$ change their signs. Using that $g_0$ is an even function and that change of sign of variables as well as parameters in the continuum function cannot change its value, one obtains

$$I_{32} = \int d^3k'_{23}\, g^*_{\mathbf{k}'^f_{23}}(\mathbf{k}'_{23})\, g_0(\mathbf{k}'_{23} + \mathbf{K}^f_{23}/2). \tag{21}$$

## IV. The antisymmetry requirement

For nucleon-deuteron scattering the incoming particle and one of the particles in the deuteron are identical, and we assume in the following that these are particles 1 and 2. The exclusion principle is easily taken into account by multiplying the transition operator $T_{1d}$ to the left by the operator $1-(12)_\sigma(12)_x$, where $(12)_\sigma$ and $(12)_x$ are permutation operators acting on the spin and position coordinates, respectively. We then get

$$T^a_{1d} = T^a_{12} + T_{13} - (12)_\sigma (12)_x T_{13} \tag{22}$$

where $T^a_{1d}$ is the nucleon-deuteron transition operator, antisymmetrized in the like particles, and

$$T^a_{12} = [1 - (12)_\sigma (12)_x]\, T_{12}$$

which is simply the antisymmetrized transition matrix for particles 1 and 2, and gives the same scattering amplitude as in free nucleon-nucleon scattering for identical nucleons. $T_{13}$ gives the scattering amplitude for two non-identical free nucleons. The term

$$-(12)_\sigma (12)_x T_{13} \tag{23}$$

represents a kind of exchange scattering, in the elastic case corresponding to so called pick-up processes. We can treat this term by the methods employed for the other terms, and get for the matrix elements in spin space

$$\begin{aligned} &(S', M', s'_{23}|(12)_\sigma (12)_x T_{13}|S, M, s) = \\ &\qquad = \sum_{s_{13}} (S', M', s'_{13}|(12)_x T_{13}|S, M, s_{13})\,(S, M, s_{13}|S, M, s) \end{aligned} \tag{24}$$

with $s'_{13} = s'_{23}$.

The complete matrix element $(f\,|\,(12)_x T_{13}\,|\,i)$ (where we do not indicate the spin explicitly) can be simplified considerably. We expand in momentum eigenfunctions and replace $(12)_x$ by $(12)_k$, the permutation operator in momentum space. Operating to the left with $(12)_k$ on the momentum coordinates we get (cf. eq. (16))

$$(\mathbf{k}_1'\,\mathbf{k}_2'\,\mathbf{k}_3'\,|\,(12)_k\,|\,f) = \delta\,(\mathbf{k}_1^f - \mathbf{k}_2')\,\delta\,(\mathbf{K}_{23}^f - \mathbf{k}_1' - \mathbf{k}_3')\,g_{\mathbf{k}_{23}^f}\left(\frac{\mathbf{k}_1' - \mathbf{k}_3'}{2}\right).$$

After simplification one gets for the matrix element

$$(f\,|\,(12)_k\,T_{13}\,|\,i) = \delta\,(\mathbf{k}_1^f + \mathbf{k}_2^f + \mathbf{k}_3^f - \mathbf{k}_1^i)\,g_0\,(\mathbf{k}_1^f)\int d^3k_{13}'\,g^*_{\mathbf{k}_{23}^f}\,(\mathbf{k}_{13}')\left(\mathbf{k}_{13}'\,|\,t_{13}\,|\,\frac{\mathbf{k}_1^i - \mathbf{k}_1^f}{2}\right). \qquad (25)$$

As before, we approximate the integral in (25) by replacing the matrix element of $t_{13}$ by a constant value. In analogy with equation (19) and (20) we obtain

$$\begin{aligned} (f\,|\,(12)_k\,T_{13}\,|\,i) &= \delta\,(\mathbf{k}_1^f + \mathbf{k}_2^f + \mathbf{k}_3^f - \mathbf{k}_1^i)\,N\,(13)\,J \\ N\,(13) &= \left(\frac{\mathbf{k}_2^f - \mathbf{k}_3^f}{2}\,|\,t_{13}\,|\,\frac{\mathbf{k}_1^i + \mathbf{k}_1^f}{2}\right) \qquad (26) \\ J &= g_0\,(\mathbf{k}_1^f)\int d^3k_{13}'\,g^*_{\mathbf{k}_{23}^f}\,(\mathbf{k}_{13}'). \end{aligned}$$

The result (25) has a close relation to the theory of pick-up processes by Chew and Goldberger [11], who, however, make use of the Born approximation. To demonstrate the similarity we note first that the integral in (25) can be written as

$$\left(\Psi_{\mathbf{k}_{23}^f}\,|\,t_{13}\,|\,\frac{\mathbf{k}_1^i + \mathbf{k}_1^f}{2}\right). \qquad (27)$$

For a real pick-up process, the continuum state to the left in (27) is replaced by the deuteron ground state $(\varphi_0|$, which satisfies the Schrödinger equation

$$(\varphi_0\,|\,V = (\varphi_0\,|\,(E_B - \mathbf{p}^2/M) \qquad (28)$$

where $V$ is the proton-neutron potential, $-E_B$ is the binding energy of the deuteron, $\mathbf{p}$ is the momentum operator, and $M$ is the nucleon mass. Momentum conservation gives

$$\begin{aligned} (\mathbf{k}_1^i + \mathbf{k}_1^f)/2 &= \mathbf{k}_1^i - (\mathbf{k}_2^f + \mathbf{k}_3^f)/2 \\ &= \mathbf{k} - \mathbf{K}/2 \end{aligned}$$

in Chew's and Goldberger's notation. In (27) we now replace $t_{13}$ by $V_{13}$, thus introducing the Born approximation, and use (28) to get

$$\left(\varphi_0\,|\,t_{13}\,|\,\frac{\mathbf{k}_1^i + \mathbf{k}_1^f}{2}\right) \simeq [E_B - (\hbar^2/M)\,(\mathbf{k} - \mathbf{K}/2)^2]\,(\varphi_0\,|\,\mathbf{k} - \mathbf{K}/2)$$

which agrees with equation (7) of reference [11].

## VII. Influence of the Coulomb interaction

The treatment given so for is strictly valid only for neutron-deuteron scattering. The influence of Coulomb scattering can easily be taken into account by a slight reformulation of the treatment given by Gell-Mann and Goldberger [8]. Utilizing their equations[1] (4.4) and (4.5) we can write

$$T_{N+C} = T_C + \Omega_C^{(-)\dagger} T_{N(C)} \Omega_C^{(+)}. \tag{29}$$

Here $T_{N+C}$ is the transition operator for combined nuclear and Coulomb scattering, $T_C$ is the transition operator for pure Coulomb scattering, and $T_{N(C)}$ is the transition operator describing transitions between different Coulomb eigenstates, i.e. the transition operator obtained if the Coulomb potential is included in the unperturbed Hamiltonian. $\Omega_C^{(\pm)}$ are the wave operators for ingoing and outgoing Coulomb waves.

In the spirit of the impulse approximation we approximate $T_{N(C)}$ by a sum of two-particle operators

$$T_{N(C)} = T_{N(C),12} + T_{N(C),13}. \tag{30}$$

If 1 and 2 are the two protons, the operator

$$T_{12} = T_C + \Omega_C^{(-)\dagger} T_{N(C),12} \Omega_C^{(+)} \tag{31}$$

is equal to the transition operator for proton-proton scattering. The operator

$$T_{C,13} = \Omega_C^{(-)\dagger} T_{N(C),13} \Omega_C^{(+)} \tag{32}$$

is the proton-neutron transition operator modified for the distortion of the incoming and the outgoing waves due to the presence of the second proton. With these reinterpretations the treatment given in the preceding sections can be applied to proton-deuteron scattering.

## VIII. Formulae for the transition matrix elements

In the preceding sections we have discussed the expansion of matrix elements of the nucleon-deuteron transition operator in spin space and in momentum space, separately. Here we wish to combine the results of these investigations in a final formula. We have then to introduce again the spin states (initial, intermediate, and final) into the equations (19), (20), (21), and (26).

For the final state we have, according to (13),

$$(S' \mid f) = (S' \mid \Omega_{23}^{(-)} \mid S^f)$$

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[1] Note the printing error in eq. (4.4)!

where, for brevity, the momentum vectors have been omitted and $S$ stands for the whole set of spin quantum numbers $S$, $M$, $s_{23}$. $S^f$ is the final spin state and $S'$ the intermediate spin state discussed in section IV. Let us, for a moment, write the initial state in a similar way:

$$(S \mid i) = (S \mid \Omega_0 \mid S^i)$$

thus introducing another intermediate state $S$ between the initial spin state $S^i$ and the transition operator.[1] The overlap integrals $I_{23}$, $I_{32}$, and $J$ will then depend on the initial, the final and both the intermediate states, so that we should regard these quantities as 4-index matrices and write, for instance, $(S^f, S \mid I_{23} \mid S', S^i)$ instead of $I_{23}$. The transition operator will appear between the intermediate states $S'$ and $S$, and summing over these we get, for instance,

$$(S^f \mid T_{12} \mid S^i) \sim \sum_{S,S'} (S^f, S \mid I_{23} \mid S', S^i)\,(S' \mid t_{12} \mid S).$$

(Cf. eq. (19); we have omitted the delta function representing the momentum conservation.) However, with our definition (11) of the initial state we have $S = S^i$, so that only the summation over the intermediate state $S'$ remains. It is therefore sufficient to interprete the quanties $I_{rs}$ and $J$ as ordinary 2-index matrices $\mathbf{I}_{rs}$ and $\mathbf{J}$, depending in general, though, on the initial state considered. On this understanding we can write for the complete transition matrix $\mathbf{M}^a\,(1\,d)$ in three-body spin space, using (22)

$$\mathbf{M}^a\,(1\,d) = \mathbf{I}_{23} \cdot \mathbf{M}^a\,(12) + \mathbf{I}_{32} \cdot \mathbf{M}\;(13) - \mathbf{J} \cdot \mathbf{N}. \tag{33}$$

Here $\mathbf{M}^a\,(1\,d)$ is defined by

$$(\mathbf{k}_1^f,\, \mathbf{k}_2^f,\, \mathbf{k}_3^f \mid \Omega_{23}^{(-)\dagger}\, T_{1d}^a \mid \mathbf{k}_1^i,\, \Phi_0) = \delta\,(\mathbf{k}_1^f + \mathbf{k}_2^f + \mathbf{k}_3^f - \mathbf{k}_1^i)\,\mathbf{M}^a\,(1\,d). \tag{34}$$

$T_{1d}^a$, $\mathbf{M}^a\,(1\,d)$, and $\mathbf{M}^a\,(12)$ are antisymmetrized in particles 1 and 2. $\mathbf{I}_{rs}$, $\mathbf{J}$, $\mathbf{M}^a\,(12)$, and $\mathbf{M}\,(13)$ are matrices in three-body spin space and depend on the momenta according to the formulae (19), (20), (21), and (26). In these formulae, $g^*_{\mathbf{k}'_{23}}\,(\mathbf{k}'_{23})$ etc. should be replaced by

$$(\mathbf{k}_{23}^f,\, S^f,\, M^f,\, s_{23}^f \mid \Omega_{23}^{(-)\dagger} \mid \mathbf{k}'_{23},\, S',\, M',\, s'_{23}) \tag{35}$$

etc., using the relation (13).

We can write (33) more explicitly as

---

[1] Actually, in the presence of tensor forces the deuteron ground state is often written (see, e.g., Sachs [12])

$$|\varphi_0) = \{u + w\,S_{23}\}\,|\,\chi)$$

where $S_{23}$ is the tensor operator for the particles 2 and 3 and $|\,\chi\,)$ a two-body (triplet) spin state. $u$ and $w$ are the radial $S$- and $D$-wave functions, respectively.

$$
\begin{aligned}
(S^f, M^f, s_{23}^f \,|\, \mathbf{M}^a(1\,d) \,|\, S^i, M^i, s_{23}^i) = \\
= \sum_{S,M} \{ & (S^f, M^f, s_{23}^f \,|\, \mathbf{I}_{23} \,|\, S, M, s_{23}^f)\,(S, M, s_{23}^f \,|\, \mathbf{M}^a(12) \,|\, S^i, M^i, s_{23}^i) + \\
& + (S^f, M^f, s_{23}^f \,|\, \mathbf{I}_{32} \,|\, S, M, s_{23}^f)\,(S, M, s_{23}^f \,|\, \mathbf{M}(13) \,|\, S^i, M^i, s_{23}^i) - \\
& - (S^f, M^f, s_{23}^f \,|\, \mathbf{J} \,|\, S, M, s_{23}^f)\,(S, M, s_{23}^f \,|\, \mathbf{N} \,|\, S^i, M^i, s_{23}^i)\}
\end{aligned}
\tag{36}
$$

where we have made use of the diagonality of $\mathbf{I}_{rs}$ and $\mathbf{J}$ in $s_{23}$.

For central forces the matrices $\mathbf{I}_{rs}$ and $\mathbf{J}$ are diagonal in and independent of $S$ and $M$. Simple estimates show that for non-central forces it seems to be a good approximation to neglect the non-diagonal parts. In this approximation (36) simplifies to

$$
\begin{aligned}
(S^f, M^f, s_{23}^f \,|\, \mathbf{M}^a(1\,d) \,|\, S^i, M^i, s_{23}^i) = \\
= I_{23}^s\,(S^f, M^f, s_{23}^f \,|\, \mathbf{M}^a(12) \,|\, S^i, M^i, s_{23}^i) + \\
+ I_{32}^s\,(S^f, M^f, s_{23}^f \,|\, \mathbf{M}(13) \,|\, S^i, M^i, s_{23}^i) - \\
- J^s\,(S^f, M^f, s_{23}^f \,|\, \mathbf{N} \,|\, S^i, M^i, s_{23}^i)
\end{aligned}
\tag{37}
$$

where

$$I_{23}^s = (S, M, s \,|\, \mathbf{I}_{23} \,|\, S, M, s) \tag{38}$$

and similarly for $I_{32}^s$ and $J^s$ with $s = s_{23}^f$.

The matrix elements of $\mathbf{M}^a(12)$, $\mathbf{M}(13)$ and $\mathbf{N}$ are transformed according to (8) and reduced to ordinary two-particle matrix elements according to (10), using the tables III and IV. Table V in the appendix gives the matrix elements of $\mathbf{M}^a(1\,d)$ calculated according to (37).

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## APPENDIX

*Table III.* Matrix elements in three-body spin space of $T_{1d} = T_{12} + T_{13}$, according to equation (8) in the text.

$$(3/2, M', 1 \mid T_{1d} \mid 3/2, M, 1) = (3/2, M', 1 \mid T_{12} \mid 3/2, M, 1) + (3/2, M', 1 \mid T_{13} \mid 3/2, M, 1)$$

$$(3/2, M', 1 \mid T_{1d} \mid 1/2, M, 0) = -\sqrt{3}/2\,(3/2, M', 1 \mid T_{12} \mid 1/2, M, 1) + \sqrt{3}/2\,(3/2, M', 1 \mid T_{13} \mid 1/2, M, 1)$$

$$(3/2, M', 1 \mid T_{1d} \mid 1/2, M, 1) = -1/2\,(3/2, M', 1 \mid T_{12} \mid 1/2, M, 1) - 1/2\,(3/2, M', 1 \mid T_{13} \mid 1/2, M, 1)$$

$$(1/2, M', 0 \mid T_{1d} \mid 3/2, M, 1) = -\sqrt{3}/2\,(1/2, M', 1 \mid T_{12} \mid 3/2, M, 1) + \sqrt{3}/2\,(1/2, M', 1 \mid T_{13} \mid 3/2, M, 1)$$

$$(1/2, M', 1 \mid T_{1d} \mid 3/2, M, 1) = -1/2\,(1/2, M', 1 \mid T_{12} \mid 3/2, M, 1) - 1/2\,(1/2, M', 1 \mid T_{13} \mid 3/2, M, 1)$$

$$\begin{aligned}(1/2, M', 0 \mid T_{1d} \mid 1/2, M, 0) = {} & 1/4\,(1/2, M', 0 \mid T_{12} \mid 1/2, M, 0) + 3/4\,(1/2, M', 1 \mid T_{12} \mid 1/2, M, 1) + \\ & + 1/4\,(1/2, M', 0 \mid T_{13} \mid 1/2, M, 0) + 3/4\,(1/2, M', 1 \mid T_{13} \mid 1/2, M, 1)\end{aligned}$$

$$\begin{aligned}(1/2, M', 0 \mid T_{1d} \mid 1/2, M, 1) = {} & -\sqrt{3}/4\,(1/2, M', 0 \mid T_{12} \mid 1/2, M, 0) + \sqrt{3}/4\,(1/2, M', 1 \mid T_{12} \mid 1/2, M, 1) + \\ & + \sqrt{3}/4\,(1/2, M', 0 \mid T_{13} \mid 1/2, M, 0) - \sqrt{3}/4\,(1/2, M', 1 \mid T_{13} \mid 1/2, M, 1)\end{aligned}$$

$(1/2, M', 1 \mid T_{1d} \mid 1/2, M, 0) = (1/2, M', 0 \mid T_{1d} \mid 1/2, M, 1) =$ preceding element.

$$\begin{aligned}(1/2, M', 1 \mid T_{1d} \mid 1/2, M, 1) = {} & 3/4\,(1/2, M', 0 \mid T_{12} \mid 1/2, M, 0) + 1/4\,(1/2, M', 1 \mid T_{12} \mid 1/2, M, 1) + \\ & + 3/4\,(1/2, M', 0 \mid T_{13} \mid 1/2, M, 0) + 1/4\,(1/2, M', 1 \mid T_{13} \mid 1/2, M, 1).\end{aligned}$$

*Table IV.* Matrix elements in three-body spin space of two-body operators $T_{1k}$, according to equation (10) in the text.

$S' = S = 3/2,\ s' = s = 1.$

$(3/2, 3/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) \quad = (1, 1 \mid T_{1k} \mid 1, 1)$

$(3/2, 3/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) \quad = \sqrt{6}/3\,(1, 1 \mid T_{1k} \mid 1, 0)$

$(3/2, 2/2, 1 \mid T_{1k} \mid 3/2, -1/2, 1) \quad = 1/\sqrt{3}\,(1, 1 \mid T_{1k} \mid 1, -1)$

$(3/2, 3/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) \quad = 0$

$(3/2, 1/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) \quad = \sqrt{6}/3\,(1, 0 \mid T_{1k} \mid 1, 1)$

$(3/2, 1/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) \quad = 2/3\,(1, 0 \mid T_{1k} \mid 1, 0) + 1/3\,(1, 1 \mid T_{1k} \mid 1, 1)$

$(3/2, 1/2, 1 \mid T_{1k} \mid 3/1, -1/2, 1) \quad = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, -1) + \sqrt{2}/3\,(1, 1 \mid T_{1k} \mid 1, 0)$

$(3/2, 1/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) \quad = 1/\sqrt{3}\,(1, 1 \mid T_{1k} \mid 1, -1)$

*Table IV.* (Contd.)

$(3/2, -1/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) = 1/\sqrt{3}\,(1, -1 \mid T_{1k} \mid 1, 1)$

$(3/2, -1/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 1) + \sqrt{2}/3\,(1, -1 \mid T_{1k} \mid 1, 0)$

$(3/2, -1/2, 1 \mid T_{1k} \mid 3/2, -1/2, 1) = 2/3\,(1, 0 \mid T_{1k} \mid 1, 0) + 1/3\,(1, -1 \mid T_{1k} \mid 1, -1)$

$(3/2, -1/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) = \sqrt{6}/3\,(1, 0 \mid T_{1k} \mid 1, -1)$

$(3/2, -3/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) = 0$

$(3/2, -3/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) = 1/\sqrt{3}\,(1, -1 \mid T_{1k} \mid 1, 1)$

$(3/2, -3/2, 1 \mid T_{1k} \mid 3/2, -1/2, 1) = \sqrt{6}/3\,(1, -1 \mid T_{1k} \mid 1, 0)$

$(3/2, -3/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) = (1, -1 \mid T_{1k} \mid 1, -1)$

### $S' = S = 1/2,\ s' = s = 0.$

$(1/2, 1/2, 0 \mid T_{1k} \mid 1/2, 1/2, 0) = (1/2, -1/2, 0 \mid T_{1k} \mid 1/2, -1/2, 0) = (0, 0 \mid T_{1k} \mid 0, 0)$

$(1/2, 1/2, 0 \mid T_{1k} \mid 1/2, -1/2, 0) = (1/2, -1/2, 0 \mid T_{1k} \mid 1/2, 1/2, 0) = 0$

### $S' = S = 1/2,\ s' = s = 1.$

$(1/2, 1/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = 1/3\,(1, 0 \mid T_{1k} \mid 1, 0) + 2/3\,(1, 1 \mid T_{1k} \mid 1, 1)$

$(1/2, 1/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, -1) + \sqrt{2}/3\,(1, 1 \mid T_{1k} \mid 1, 0)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 1) + \sqrt{2}/3\,(1, -1 \mid T_{1k} \mid 1, 0)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = 1/3\,(1, 0 \mid T_{1k} \mid 1, 0) + 2/3\,(1, -1 \mid T_{1k} \mid 1, -1)$

### $S' = 3/2,\ S = 1/2,\ s' = s = 1.$

$(3/2, 3/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = -1/\sqrt{3}\,(1, 1 \mid T_{1k} \mid 1, 0)$

$(3/2, 3/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = -\sqrt{6}/3\,(1, 1 \mid T_{1k} \mid 1, -1)$

$(3/2, 1/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = -\sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 0) + \sqrt{2}/3\,(1, 1 \mid T_{1k} \mid 1, 1)$

$(3/2, 1/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = -2/3\,(1, 0 \mid T_{1k} \mid 1, -1) + 1/3\,(1, 1 \mid T_{1k} \mid 1, 0)$

$(3/2, -1/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = 2/3\,(1, 0 \mid T_{1k} \mid 1, 1) - 1/3\,(1, -1 \mid T_{1k} \mid 1, 0)$

$(3/2, -1/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 0) - \sqrt{2}/3\,(1, -1 \mid T_{1k} \mid 1, -1)$

$(3/2, -3/2, 1 \mid T_{1k} \mid 1/2, 1/2, 1) = \sqrt{6}/3\,(1, -1 \mid T_{1k} \mid 1, 1)$

$(3/2, -3/2, 1 \mid T_{1k} \mid 1/2, -1/2, 1) = 1/\sqrt{3}\,(1, -1 \mid T_{1k} \mid 1, 0)$

*Table IV.* (Contd.)

$S' = 1/2,\ S = 3/2,\ s' = s = 1.$

$(1/2, 1/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) = -1/\sqrt{3}\,(1, 0 \mid T_{1k} \mid 1, 1)$

$(1/2, 1/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) = -\sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 0) + \sqrt{2}/3\,(1, 1 \mid T_{1k} \mid 1, 1)$

$(1/2, 1/2, 1 \mid T_{1k} \mid 3/2, -1/2, 1) = 2/3\,(1, 1 \mid T_{1k} \mid 1, 0) - 1/3\,(1, 0 \mid T_{1k} \mid 1, -1)$

$(1/2, 1/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) = \sqrt{6}/3\,(1, 1 \mid T_{1k} \mid 1, -1)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 3/2, 3/2, 1) = -\sqrt{6}/3\,(1, -1 \mid T_{1k} \mid 1, 1)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 3/2, 1/2, 1) = -2/3\,(1, -1 \mid T_{1k} \mid 1, 0) + 1/3\,(1, 0 \mid T_{1k} \mid 1, 1)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 3/2, -1/2, 1) = \sqrt{2}/3\,(1, 0 \mid T_{1k} \mid 1, 0) - \sqrt{2}/3\,(1, -1 \mid T_{1k} \mid 1, -1)$

$(1/2, -1/2, 1 \mid T_{1k} \mid 3/2, -3/2, 1) = 1/\sqrt{3}\,(1, 0 \mid T_{1k} \mid 1, -1)$

*Table V.* Matrix elements in three-body spin space of $\mathbf{M}^a(1\,d)$, according to equation (37) in the text.

In this table, $M^a_{m'm} = (1, m' \mid \mathbf{M}^a(12) \mid 1, m)$, $M^a_{ss} = (0, 0 \mid \mathbf{M}^a(12) \mid 0, 0)$, and similarly for $\mathbf{M}(13)$ and $\mathbf{N}$. The quantities $I^s_{23}$, etc., are defined in equation (38). To save space, those matrix elements are omitted for which expressions can be derived from those given by changing the sign of all projection quantum numbers and multiplying by the factor $F$ given in the headline above each group of matrix elements.

$S^f = S^i = 3/2,\ s^f_{23} = 1.\ F = +1.$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, 3/2, 1) = I^1_{23} M^a_{11} + I^1_{32} M_{11} - J^1 N_{11}$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, 1/2, 1) = \sqrt{6}/3\,(I^1_{23} M^a_{10} + I^1_{32} M_{10} - J^1 N_{10})$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, -1/2, 1) = 1/\sqrt{3}\,(I^1_{23} M^a_{1-1} + I^1_{32} M_{1-1} - J^1 N_{1-1})$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, -3/2, 1) = 0$

$(3/2, 1/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, 3/2, 1) = \sqrt{6}/3\,(I^1_{23} M^a_{01} + I^1_{32} M_{01} - J^1 N_{01})$

$(3/2, 1/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, 1/2, 1) = 1/3\,[I^1_{23}\,(2\,M^a_{00} + M^a_{11}) + I^1_{32}\,(2\,M_{00} + M_{11}) - J^1\,(2\,N_{00} + N_{11})]$

$(3/2, 1/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, -1/2, 1) = \sqrt{2}/3\,[I^1_{23}\,(M^a_{0-1} + M^a_{10}) + I^1_{32}\,(M_{0-1} + M_{10}) - J^1\,(N_{0-1} + N_{10})]$

$(3/2, 1/2, 1 \mid \mathbf{M}^a(1\,d) \mid 3/2, -3/2, 1) = 1/\sqrt{3}\,(I^1_{23} M^a_{1-1} + I^1_{32} M_{1-1} - J^1 N_{1-1})$

$S^f = 3/2,\ S^i = 1/2,\ s^f_{23} = 1.\ F = -1.$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 1/2, 1/2, 1) = 1/2\sqrt{3}\,(I^1_{23} M^a_{10} + I^1_{32} M_{10} - J^1 N_{10})$

$(3/2, 3/2, 1 \mid \mathbf{M}^a(1\,d) \mid 1/2, -1/2, 1) = 1/\sqrt{6}\,(I^1_{23} M^a_{1-1} + I^1_{32} M_{1-1} - J^1 N_{1-1})$

*Table V.* (Contd.)

$$(3/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, 1/2, 1) = 1/3\sqrt{2}\,[I^1_{23}(M^a_{00} - M^a_{11}) + I^1_{32}(M_{00} - M_{11}) - J^1 (N_{00} - N_{11})]$$

$$\begin{aligned}(3/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, -1/2, 1) = 1/6\,[I^1_{23}(2\,M^a_{0-1} - M^a_{10}) + I^1_{32}(2\,M_{0-1} - M_{10}) - \\ - J^1 (2\,N_{0-1} - N_{10})]\end{aligned}$$

$S^f = 1/2,\ S^i = 3/2,\ s^f_{23} = 0.\ \ F = -1.$

$$(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, 3/2, 1) = 1/2\,(I^0_{23} M^a_{01} - I^0_{32} M_{01})$$

$$(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, 1/2, 1) = 1/\sqrt{6}\,[I^0_{23}(M^a_{00} - M^a_{11}) - I^0_{32}(M_{00} - M_{11})]$$

$$(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, -1/2, 1) = -1/2\sqrt{3}\,[I^0_{23}(2\,M^a_{10} - M^a_{0-1}) - I^0_{32}(2\,M_{10} - M_{0-1})]$$

$$(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, -3/2, 1) = -1/\sqrt{2}\,(I^0_{23} M^a_{1-1} - I^0_{32} M_{1-1})$$

$S^f = 1/2,\ S^i = 3/2,\ s^f_{23} = 1.\ \ F = -1.$

$$(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, 3/2, 1) = 1/2\sqrt{3}\,(I^1_{23} M^a_{01} + I^1_{32} M_{01}) + 1/\sqrt{3}\,J^1 N_{01}$$

$$\begin{aligned}(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, 1/2, 1) = 1/3\sqrt{2}\,[I^1_{23}(M^a_{00} - M^a_{11}) + I^1_{32}(M_{00} - M_{11})] + \\ + \sqrt{2}/3\,J^1 (N_{00} - N_{11})\end{aligned}$$

$$\begin{aligned}(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, -1/2, 1) = -1/6\,[I^1_{23}(2\,M^a_{10} - M^a_{0-1}) + I^1_{32}(2\,M_{10} - M_{0-1})] \\ - 1/3\,J^1 (2\,N_{10} - N_{0-1})\end{aligned}$$

$$(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 3/2, -3/2, 1) = -1/\sqrt{6}\,(I^1_{23} M^a_{1-1} + I^1_{32} M_{1-1}) - \sqrt{6}/3\,J^1 N_{1-1}$$

$S^f = S^i = 1/2,\ s^f_{23} = 0.\ \ F = +1.$

$$\begin{aligned}(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, 1/2, 1) = -1/4\sqrt{3}\,[I^0_{23}(3\,M^a_{ss} - M^a_{00} - 2\,M^a_{11}) - \\ - I^0_{32}(3\,M_{ss} - M_{00} - 2\,M_{11})] - \sqrt{3}/2\,J^0 N_{ss}\end{aligned}$$

$$(1/2, 1/2, 0 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, -1/2, 1) = -1/2\sqrt{6}\,[I^0_{23}(M^a_{0-1} + M^a_{10}) - I^0_{32}(M_{0-1} + M_{10})]$$

$S^f = S^i = 1/2,\ s^f_{23} = 1.\ \ F = +1.$

$$\begin{aligned}(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, 1/2, 1) = 1/12\,[I^1_{23}(9\,M^a_{ss} + M^a_{00} + 2\,M^a_{11}) + \\ + I^1_{32}(9\,M_{ss} + M_{00} + 2\,M_{11})] + 1/6\,J^1 (N_{00} + 2\,N_{11})\end{aligned}$$

$$\begin{aligned}(1/2, 1/2, 1 \,|\, \mathbf{M}^a (1\,d) \,|\, 1/2, -1/2, 1) = \sqrt{2}/12\,[I^1_{23}(M^a_{0-1} + M^a_{10}) + I^1_{32}(M_{0-1} + M_{10})] + \\ + 1/3\sqrt{2}\,J^1 (N_{0-1} + N_{10})\end{aligned}$$

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